# 15.5 Lecture 2: Finding bounds for 3D regions

Jeremiah Southwick
(and some from Robert Vandermolen)

Spring 2019

# Links

Robert's slides can be found here:

`http://people.math.sc.edu/robertv/teaching.html`

The 15.5 slides can be found here:

`https://docs.google.com/presentation/d/`
`1r8MEhvziyxwyQ1DTMYaUsiz90CzZTemKe040LI6K8Cc`

# Last class

Process for finding limits in *dzdydx* order (pg. 907-909)

1. Sketch $D$ and its shadow in the $xy$-plane.
2. Find $z$-limits of integration (the top and the bottom of the region).

$$f_1(x,y) \leq z \leq f_2(x,y)$$

3. Find $y$-limits of integration.

$$g_1(x) \leq y \leq g_2(x)$$

4. Find $x$-limits of integration.

$$a \leq x \leq b$$

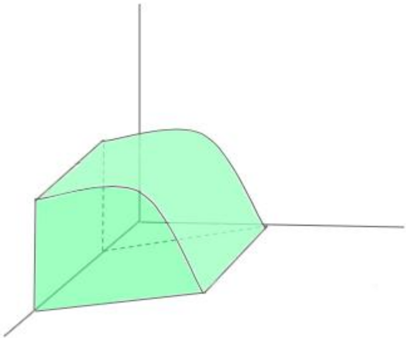
$$\int\int\int_D f(x,y,z)dV = \int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

## Triple Integration!

Now, we will find it helpful later to switch the order of integration as we did with double integrals, so let's practice finding appropriate bounds for our integration.

### Example:

Set up the triple integral for the volume of the region:



$R:$

$x \geq 0,\ y \geq 0,\ z \geq 0$

bounded above by the cylinder: $z = 1 - y^2$

between the vertical planes $x + y = 1$ and $x + y = 3$

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$R:$

$x \geq 0,\ y \geq 0,\ z \geq 0$

bounded above by the cylinder: $z = 1 - y^2$

between the vertical planes $x + y = 1$ and $x + y = 3$

First, let's pick the order: dz dy dx

since z has the bounds:

$$0 \leq z \leq 1 - y^2$$

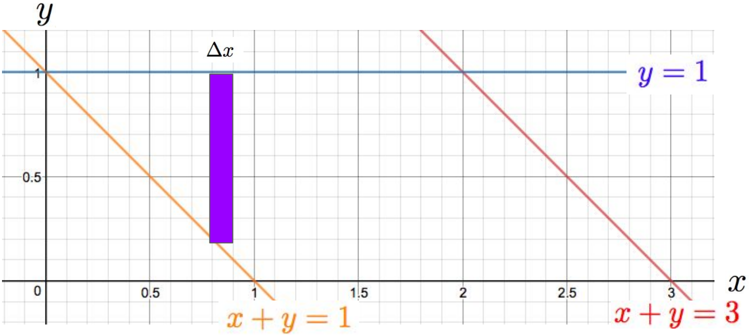
we can begin by setting up the integral:

$$\int_?^? \int_?^? \int_0^{1-y^2} dz\, dy\, dx$$

So next, we should look at what is happening in the xy-plane, to determine the remaining bounds...

## Triple Integration!

## Example:




$R$ :

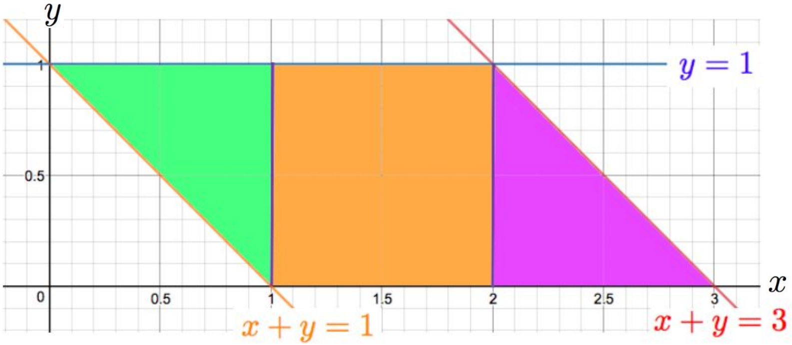
$x \geq 0,\ y \geq 0,\ z \geq 0$

bounded above by the cylinder: $z = 1 - y^2$

between the vertical planes $x + y = 1$ and $x + y = 3$

## Triple Integration!

### Example:




$$\int_0^1 \int_{1-x}^1 \int_0^{1-z^2} dz\, dy\, dx + \int_1^2 \int_0^1 \int_0^{1-y^2} dz\, dy\, dx + \int_2^3 \int_0^{3-x} \int_0^{1-y^2} dz\, dy\, dx$$

## TRIPLE INTEGRATION!

Now, we will find it helpful later to switch the order of integration as we did with double integrals, so let's practice finding appropriate bounds for our integration.

### EXAMPLE:

Set up the triple integral for the volume of the region:

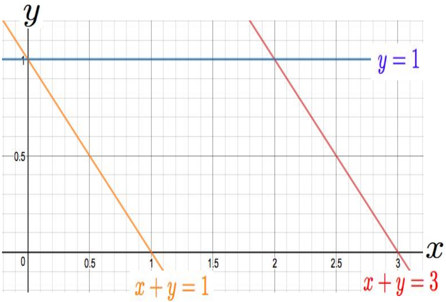
$R:$

$x \geq 0, \; y \geq 0, \; z \geq 0$

bounded above by the cylinder: $z = 1 - y^2$

between the vertical planes $x + y = 1$ and $x + y = 3$

Now YOU TRY the order: dz dx dy




$$\int_0^1 \int_{1-y}^{3-y} \int_0^{1-y^2} dz\, dx\, dy$$

## Triple Integration!

Now, we will find it helpful later to switch the order of integration as we did with double integrals, so let's practice finding appropriate bounds for our integration.

### Example:

Set up the triple integral for the volume of the region:

$R:$

$x \geq 0,\ y \geq 0,\ z \geq 0$

bounded above by the cylinder: $z = 1 - y^2$

between the vertical planes $x + y = 1$ and $x + y = 3$

Now, let's try a tricker order: dx dy dz

since x has the bounds:

$$1 - y \leq x \leq 3 - y$$

we can begin by setting up the integral:

$$\int_{?}^{?} \int_{?}^{?} \int_{1-y}^{3-y} dx\, dy\, dz$$

The tricky part is that now we should look at the projection (the shadow) in the yz-plane, to determine the remaining bounds...

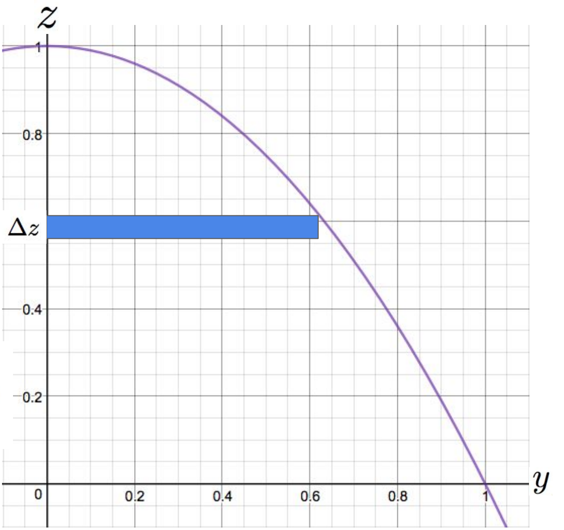
## Triple Integration!

Now, we will find it helpful later to switch the order of integration as we did with double integrals, so let's practice finding appropriate bounds for our integration.

### Example:

Set up the triple integral for the volume of the region:

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_{1-y}^{3-y} dx\, dy\, dz$$



# Example

## Example

*Rewrite the following integral using the order dydzdx.*

$$\int_{x=0}^{x=4} \int_{y=0}^{y=(4-x)/2} \int_{z=0}^{z=(12-3x-y)/4} dz\ dy\ dx$$

# Top and bottom

## Example

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$$\int_{x=0}^{x=4} \int_{y=0}^{y=(4-x)/2} \int_{z=0}^{z=(12-3x-y)/4} dz\ dy\ dx$$

In the integral, $z$ is bounded between $z = 0$ and $z = (12 - 3x - y)/4$.

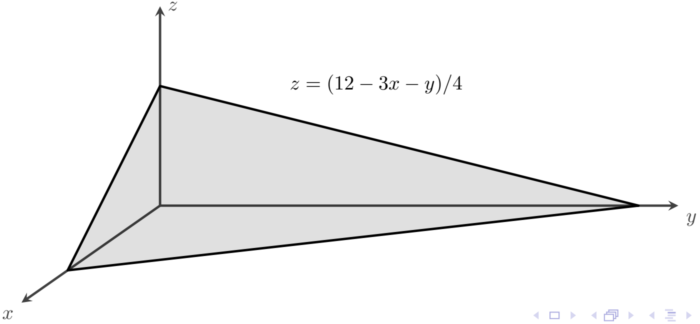
# Top and bottom

## Example

*Rewrite the following integral using the order dydzdx.*

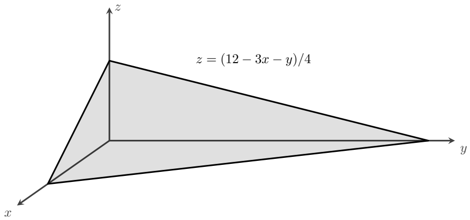
$$\int_{x=0}^{x=4} \int_{y=0}^{y=(4-x)/2} \int_{z=0}^{z=(12-3x-y)/4} dz\ dy\ dx$$

In the integral, $z$ is bounded between $z = 0$ and $z = (12 - 3x - y)/4$. This is the plane pictured below.



# Shadow

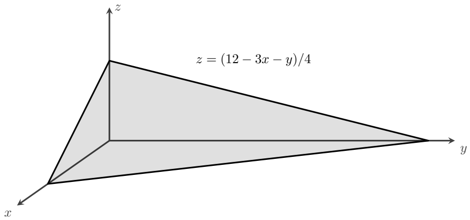
$$\int_{x=0}^{x=4} \int_{y=0}^{y=(4-x)/2} \int_{z=0}^{z=(12-3x-y)/4} \; dz \; dy \; dx$$




We don't want all of this region, but only the portion that is satisfied by the bounds on $x$ and $y$ in the integral. Thus we consider the shadow described by the bounds in the integral:

# Shadow

$$\int_{x=0}^{x=4} \int_{y=0}^{y=(4-x)/2} \int_{z=0}^{z=(12-3x-y)/4} \; dz \; dy \; dx$$



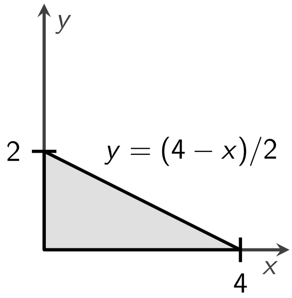

We don't want all of this region, but only the portion that is satisfied by the bounds on $x$ and $y$ in the integral. Thus we consider the shadow described by the bounds in the integral:

$$0 \leq y \leq (4-x)/2 \;\; \text{and} \;\; 0 \leq x \leq 4$$

# Shadow

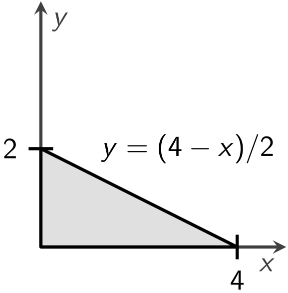
$$0 \le y \le (4 - x)/2 \quad \text{and} \quad 0 \le x \le 4$$

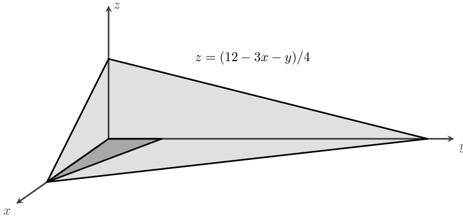
These inequalities describe the triangular region shown below.



# Shadow

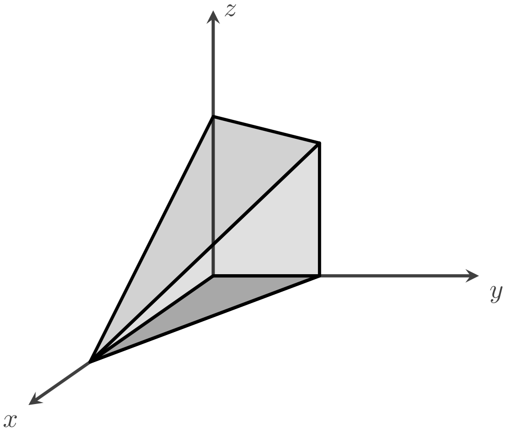
The actual region is the region between $z = (12 - 3x - y)/4$ and the $xy$-plane, but only over the shadow.



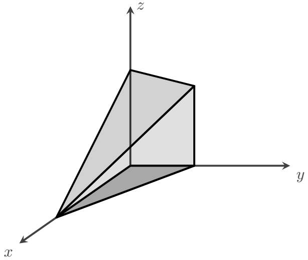



# Region of integration

The actual region of integration is pictured below.

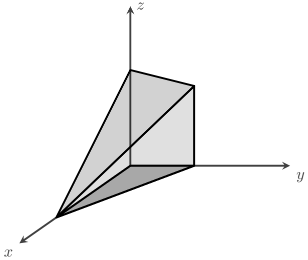


# Bounds in order $dydzdx$




If we want to move the $y$-bounds to the inside, we need to think about what the top and bottom of the region looks like in the $y$-direction.

# Bounds in order *dydzdx*

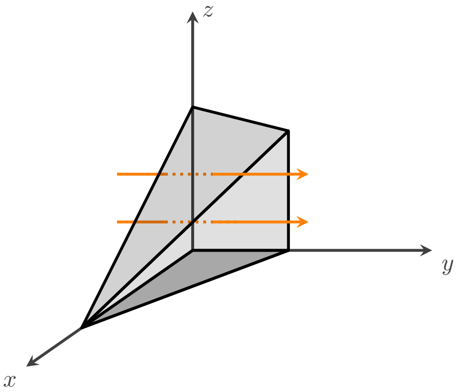



If we want to move the $y$-bounds to the inside, we need to think about what the top and bottom of the region looks like in the $y$-direction.

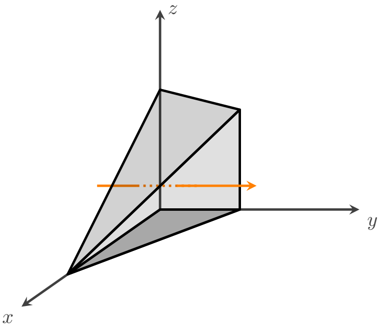
$y$ is bounded below by $y = 0$ over the whole region.

# Bounds in order *dydzdx*

The upper bound on $y$ is a bit more complicated. There are two kinds of rays that could pass through the region in the $y$-direction: First, there are rays that leave on the vertical plane $y = (4 - x)/2$. Second, there are rays that leave on the plane $z = (12 - 3x - y)/4$. Both kinds are pictured below.

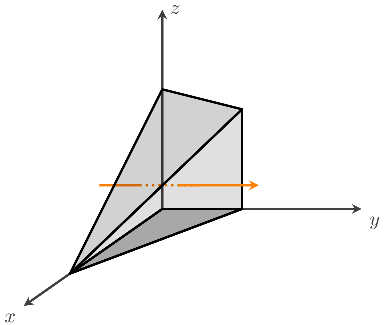


# First kind of ray




For the first kind of ray, we have $y$ bounded above by the vertical plane $y = (4 - x)/2$.

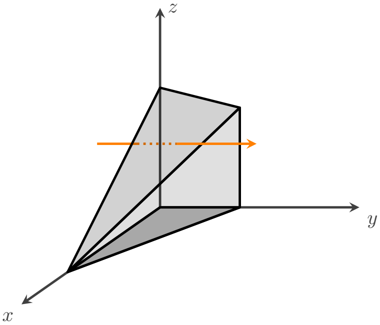
# First kind of ray




For the first kind of ray, we have $y$ bounded above by the vertical plane $y = (4 - x)/2$.

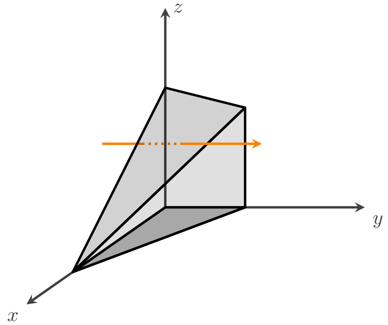
$$0 \leq y \leq (4 - x)/2$$

# Second kind of ray




For the second kind of ray, we have $y$ bounded above by the plane $z = (12 - 3x - y)/4$. Solving this for $y$, we obtain the bounds

# Second kind of ray

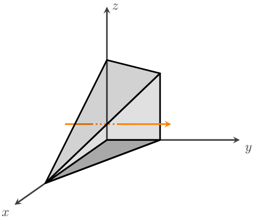


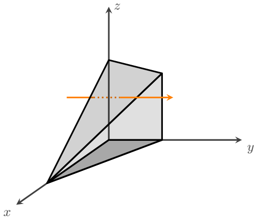

For the second kind of ray, we have $y$ bounded above by the plane $z = (12 - 3x - y)/4$. Solving this for $y$, we obtain the bounds

$$0 \leq y \leq 12 - 4z - 3x$$

# Shadows

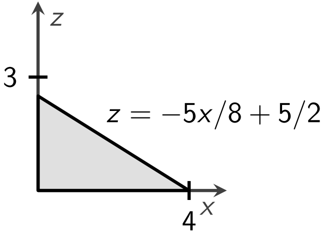
We project the shadow of each region into the $xz$-plane.

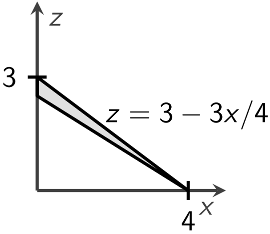



$0 \leq y \leq (4-x)/2$

$0 \leq y \leq 12 - 4z - 3x$

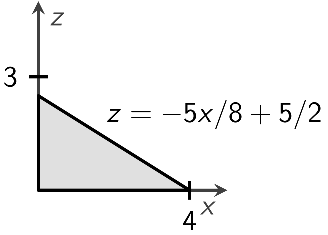


# Shadows

These shadows have the bounds shown below.

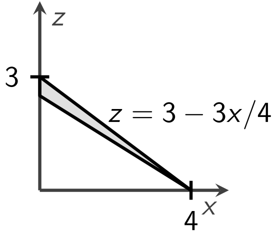
$$0 \le y \le (4 - x)/2$$




$$0 \le z \le -5x/8 + 5/2$$

$$0 \le x \le 4$$

$$0 \le y \le 12 - 4z - 3x$$




$$-5x/8 + 5/2 \le z \le 3 - 3x/4$$

$$0 \le x \le 4$$

# Example

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*Rewrite the following integral using the order dydzdx.*

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Based on the bounds we found, we can rewrite the triple integral as the sum of integrals below.

$$\int_{x=0}^{x=4} \int_{y=0}^{y=-5x/8+5/2} \int_{z=0}^{z=(4-x)/2} dz\ dy\ dx$$

$$+\int_{x=0}^{x=4} \int_{z=-5x/8+5/2}^{z=3-3x/4} \int_{y=0}^{y=12-4z-3x} dy\ dz\ dx$$

## Now you Try!

Sketch the solid whose volume is given by the following integrals, and rewrite the integral using the indicated order of integration:

- $$\int_0^4 \int_0^{(4-x)/2} \int_0^{(12-3x-y)/4} dz\, dy\, dx$$
  Rewrite using the order $dy\, dx\, dz$
- $$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{6-x-y} dz\, dy\, dx$$
  Rewrite using the order $dz\, dx\, dy$